Indian Statistical Institute, Bangalore
B.Math (Hons.) III Year/ M.Math II year 2018-2019
Semester I : Probability III/ Markov Chains

Backpaper Exam | Date: 04.01.2019
Maximum Marks: 100 | Duration: 2 hours

Note: State the results very clearly that you are using in your answers. All questions carry equal marks.

1. (a) P is a transition probability matrix on a state space of finite size. Show that P^n is also a transition probability matrix for all $n \geq 2$.

   (b) If π is a stationary distribution for P. Show that it is also a stationary distribution for P^2.

   (c) Is the converse of (b) above true?

2. Let $\{N(0,t] : t > 0\}$ be a time-homogeneous Poisson process with rate $\lambda > 0$.

   (a) Prove that $\lim_{s \to 0} \frac{P\left[N(0,t+s]-N(0,t] \geq 2\right]}{s^2}$ exists and find the value of this limit.

   (b) Compute $P\left[N(0,k] = k, \ \forall\, k = 1, 2, \ldots, n\right]$ and find its limit as $n \to \infty$.

3. Let $\{X_n\}$ be a Markov chain with state space $S = \{0\} \cup \mathbb{N}$. Suppose $P_{i,i+1} + P_{i,0} = 1$ for all $i \geq 0$ and $P_{i,i+1}$ ($= p$ say, with $0 < p < 1$) is independent of i. Classify the states of this Markov Chain. If the chain starts at state 0, find the probability of first return to 0 in exactly n steps. Find the expected time taken for the first return to state 0.

4. Given that $\{X_n\}$ is a homogeneous Markov chain, then prove that $P\left[\ X_{n-2} = i_{n-2}, X_{n-1} = i_{n-1} | X_n = i_n, X_{n+1} = i_{n+1}\right]$ is the same as $P\left[\ X_{n-2} = i_{n-2}, X_{n-1} = i_{n-1} | X_n = i_n\right]$. Find this conditional probability in terms of the transition matrix and the distribution of X_{n-2}.